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Optimal Midcourse Guidance Law for Fixed-Interval Propulsive Maneuvers

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In this paper we develop an optimal midcourse strategy for guiding an interceptor. We assume that the target is outside of the sensible atmosphere and is not maneuvering throughout the engagement and the thrust vector controlled interceptor has a fixed maneuvering time that ends well before the actual intercept. The existence of an unguided phase before the target intercept distinguishes our work from other formulations found in the literature. We also extend the optimal solution to a family of guidance laws that guarantee perfect intercept for the fixed-interval maneuvering problems.

I. Introduction

IN this paper we concentrate on the problem of guiding a missile to intercept a nonmaneuvering target moving at high velocities outside the atmosphere. The important distinguishing constraint that we impose on this problem is that the thrust vector controlled interceptor has a fixed maneuvering time that ends well before the actual intercept. Therefore, there is a period of time before the intercept that the missile is not maneuvering at all and is coasting ballistically toward the target.

This scenario is representative of a midcourse strategy during which the missile is guided so as to assure a proper collision course to the target before the depletion of the propulsive subsystem that provides the missile maneuvering capability. Following propulsion depletion, the missile coasts ballistically to the target intercept region where a small kill vehicle (KKV) is released to achieve target impact with minimum steering effort.

Here, we ignore the kill vehicle and concentrate our effort on developing the guidance law for the propelled stage that delivers the KKV to its intercept region. It turns out that the optimal guidance law for a simplified (but realistic) formulation of our problem has a closed-form solution that is an intricate modification of the proportional navigation (PN) guidance law. We will analyze the many interesting properties of this guidance law and will illustrate its performance using a computer simulation of the engagement scenario.

Note that many modern formulations of PN found in the literature[1,2] do not include a coasting phase before the impact, and the guidance law developed in this paper differs drastically from PN in this regard, and this is the main contribution of our article.

II. Optimal Guidance Law

Assuming the intercept takes place outside of any appreciable atmosphere and the target is not maneuvering, the equations of motion of the missile and the target in an appropriate inertial frame can be described as follows:

$$\dot{\boldsymbol{v}}_M = \boldsymbol{a}_M + \boldsymbol{g}_M, \qquad \dot{\boldsymbol{r}}_M = \boldsymbol{v}_M \tag{1}$$

$$\dot{\boldsymbol{v}}_T = \boldsymbol{g}_T, \qquad \dot{\boldsymbol{r}}_T = \boldsymbol{v}_T \tag{2}$$

where $\boldsymbol{r}_M$ and $\boldsymbol{r}_T$ denote the position of the missile and the target and $\boldsymbol{v}_M$ and $\boldsymbol{v}_T$ denote the respective velocities. Also, $\boldsymbol{a}_M$ represents the thrust acceleration of the missile. The two position-dependent terms $\boldsymbol{g}_M$ and $\boldsymbol{g}_T$ denote the gravitational acceleration of the missile and the target, respectively. Moreover, we assume the position difference between the target and the missile is small enough so the gravitational acceleration terms are approximately equal, $\boldsymbol{g}_M \approx \boldsymbol{g}_T$.

By subtracting Eq. (1) from Eq. (2), the relative equations of motion in terms of the relative position $\boldsymbol{r} = \boldsymbol{r}_T - \boldsymbol{r}_M$ and relative velocity $\boldsymbol{v} = \boldsymbol{v}_T - \boldsymbol{v}_M$ are as follows:

$$\dot{\boldsymbol{v}}(t) = -\boldsymbol{a}_M(t) \tag{3}$$

$$\dot{\boldsymbol{r}}(t) = \boldsymbol{v}(t) \tag{4}$$

Note that the direction of $\boldsymbol{r}$ is along the line of sight (LOS) from the missile to the target.

Our objective is to compute the missile acceleration $\boldsymbol{a}_M$ at the present time t as a function of present relative position $\boldsymbol{r}(t)$ and relative velocity $\boldsymbol{v}(t)$ so a minimum effort intercept occurs at time T that is after the missile burnout time t_f ($t_f \leq t$). To solve this problem, we compute $\boldsymbol{a}_M(t)$ by minimizing the following objective function:

$$J = \frac{\gamma}{2}\boldsymbol{r}^T(T)\boldsymbol{r}(T) + \frac{1}{2}\int_t^{t_f} \boldsymbol{a}_M^T(\tau)\boldsymbol{a}_M(\tau)\,\mathrm{d}\tau \tag{5}$$

with the weighting $\gamma \geq 0$. Note that the first term on the right-hand side of Eq. (5) is the weighted miss distance squared, and by choosing very large values for γ, we can guarantee that $\boldsymbol{r}(T)$ takes small values and an intercept at time T actually occurs.

Assuming the thrust acceleration of the missile is zero during the time period t_f to T, we have

$$\boldsymbol{r}(T) = \boldsymbol{r}(t_f) + \boldsymbol{v}(t_f)(T - t_f) \tag{6}$$

Substituting Eq. (6) in Eq. (5), the objective function J can be written as

$$J = \frac{\gamma}{2}\left[\boldsymbol{r}^T(t_f)\boldsymbol{r}(t_f) + 2\delta\boldsymbol{r}^T(t_f)\boldsymbol{v}(t_f) + \delta^2\boldsymbol{v}^T(t_f)\boldsymbol{v}(t_f)\right] + \frac{1}{2}\int_t^{t_f} \boldsymbol{a}_M^T(\tau)\boldsymbol{a}_M(\tau)\,\mathrm{d}\tau \tag{7}$$

where $\delta = T - t_f$. This objective function is a special case of the objective function

$$J = \frac{1}{2}\left[c_1\boldsymbol{v}^T(t_f)\boldsymbol{v}(t_f) + 2c_3\boldsymbol{v}^T(t_f)\boldsymbol{r}(t_f) + c_2\boldsymbol{r}^T(t_f)\boldsymbol{r}(t_f)\right] + \frac{1}{2}\int_t^{t_f} \boldsymbol{a}_M^T(\tau)\boldsymbol{a}_M(\tau)\,\mathrm{d}\tau \tag{8}$$

with c_1, c_2, and c_3 constrained to take the following values:

$$c_1 = \gamma\delta^2 \tag{9}$$

$$c_2 = \gamma \tag{10}$$

$$c_3 = \gamma\delta \tag{11}$$

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For the time being, we concentrate on minimizing Eq. (8) and assume $c_1 \geq 0, c_2 \geq 0$, and $|c_3| \leq \sqrt{c_1 c_2}$ to guarantee the nonnegativeness of the objective function. Later we will specialize the solution to the particular values of c_1, c_2, and c_3 given in Eqs. (9–11).

Note that if we set $c_3 = 0$ in Eq. (8) and constrain the problem only to one dimension, then the guidance problem discussed in Example 2 (Chapter 5) of Ref. 3 will be obtained. However, the cross term is very important to our particular application, as should be clear by looking at Eq. (7) (also see p. 287 of Ref. 3).

The linear optimal control problem (8) can be solved in closed form using standard procedures.[3] The optimal acceleration command $\boldsymbol{a}_{MO}$ has the form of state feedback

$$\boldsymbol{a}_{MO} = k_v(t)\boldsymbol{v}(t) + k_r(t)\boldsymbol{r}(t) \tag{12}$$

with the time-varying gains given by

$$k_v(t) = n_v(t_b)/d(t_b) \tag{13}$$

$$k_r(t) = n_r(t_b)/d(t_b) \tag{14}$$

where $t_b = t_f - t$ denotes the time to missile burnout and

$$n_v(t_b) = \left(1 - c_3 t_b^2/2 - c_2 t_b^3/6\right)(c_1 + c_3 t_b) + \left(t_b + c_1 t_b^2/2 + c_3 t_b^3/6\right)(c_3 + c_2 t_b) \tag{15}$$

$$n_r(t_b) = \left(1 - c_3 t_b^2/2 - c_2 t_b^3/6\right)c_3 + \left(t_b + c_1 t_b^2/2 + c_3 t_b^3/6\right)c_2 \tag{16}$$

$$d(t_b) = \left(1 + c_1 t_b + c_3 t_b^2/2\right)\left(1 - c_3 t_b^2/2 - c_2 t_b^3/6\right) + \left(t_b + c_1 t_b^2/2 + c_3 t_b^3/6\right)\left(c_3 t_b + c_2 t_b^2/2\right) \tag{17}$$

Now if we substitute Eqs. (9–11) in Eqs. (15–17), we obtain the relations

$$k_v(t) = \frac{(\delta + t_b)^2}{1/\gamma + t_b\left(\delta^2 + \delta t_b + t_b^2/3\right)} \tag{18}$$

$$k_r(t) = \frac{(\delta + t_b)}{1/\gamma + t_b\left(\delta^2 + \delta t_b + t_b^2/3\right)} \tag{19}$$

for the optimal gains that minimize the objective function given in Eq. (7). Note that the limits of k_v and k_r as γ approaches infinity are well defined and are obtained by simply setting $1/\gamma = 0$ in Eqs. (18) and (19). We will discuss this issue in more detail in the next section.

Using the definition of δ and t_b, we have $\delta + t_b = T - t$. The term $T - t$ is usually called the time to go until intercept and is denoted by t_g. Using this definition of t_g and the gain values given in Eqs. (18) and (19), the acceleration command vector can be written as

$$\boldsymbol{a}_{MO}(t) = \frac{t_g}{1/\gamma + t_b\left(t_g^2 - t_g t_b + t_b^2/3\right)}[\boldsymbol{r}(t) + t_g\boldsymbol{v}(t)] \tag{20}$$

Note that if we set $t_g = t_b$ $(T = t_f)$, which then allows maneuvering the missile until the intercept point, and let $\gamma \to \infty$ in Eq. (20), the usual proportional navigation guidance law with the navigation constant of 3 (cf. Ref. 2) will be obtained. Moreover, the zero-effort miss vector (cf. Ref. 2) $\boldsymbol{m}(t)$, which is defined as

$$\boldsymbol{m}(t) = \boldsymbol{r}(t) + t_g\boldsymbol{v}(t) \tag{21}$$

appears inside the bracket in Eq. (20). This vector determines the direction of the acceleration command vector and does not depend on t_b, so the instantaneous direction of the thrust acceleration in this guidance law is the same as the direction of thrust in proportional navigation and the two guidance laws only differ in the magnitude of the acceleration command vector.

III. Analysis of Solution

In this section we completely analyze the properties of the guidance law given in Eq. (20) for the case $\gamma \to \infty$. Specifically, we shall show in this case that the term inside the bracket shrinks at such a rate that the acceleration command $\boldsymbol{a}_{MO}$ always remains finite in spite of the fact that the coefficient outside the bracket in Eq. (20) goes to infinity as t_b approaches zero.

For this we solve the differential equation governing $\boldsymbol{r}(t)$. Let $1/\gamma = 0$ in Eq. (20) and substitute the resulting $\boldsymbol{a}_{MO}$ in Eq. (3). Then we have

$$\ddot{\boldsymbol{r}}(t) + \frac{t_g^2}{t_b\left(t_g^2 - t_g t_b + t_b^2/3\right)}\dot{\boldsymbol{r}}(t) + \frac{t_g}{t_b\left(t_g^2 - t_g t_b + t_b^2/3\right)}\boldsymbol{r}(t) = \boldsymbol{0}, \qquad t \leq t_f \tag{22}$$

where $\boldsymbol{0}$ denotes the zero vector. Note that $t_g = T - t$ and $t_b = t_f - t$, so the differential equation (22) is time varying and has a regular singular point (cf. Ref. 4) at $t = t_f$. It is interesting that this differential equation has a simple closed-form solution. It can be shown that the solution has the form

$$\boldsymbol{r}(t) = \boldsymbol{p}_1\left(t_b^3 + 3\delta t_b^2\right) + \boldsymbol{p}_2(t_b + \delta), \qquad t \leq t_f \tag{23}$$

where $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are the constant vectors determined from the initial conditions and δ is equal to $T - t_f$ (which was defined previously). Let us take the time origin ($t = 0$) as the beginning of the engagement and denote the relative position and velocity at this reference point by $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$, respectively. Then $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ in terms of $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$ are as follows:

$$\boldsymbol{p}_1 = \frac{-1}{2(T^3 - \delta^3)}\boldsymbol{m}_0 \tag{24}$$

$$\boldsymbol{p}_2 = \frac{3(T^2 - \delta^2)}{2(T^3 - \delta^3)}\boldsymbol{m}_0 - \boldsymbol{v}_0 \tag{25}$$

where $\boldsymbol{m}_0 = \boldsymbol{r}_0 + T\boldsymbol{v}_0$ denotes the zero-effort miss vector $\boldsymbol{m}(t)$ [defined in Eq. (21)] at time $t = 0$. Differentiating Eq. (23) twice and substituting for $\boldsymbol{p}_1$ from Eq. (24), we have

$$\boldsymbol{a}_{MO}(t) = -\ddot{\boldsymbol{r}}(t) = \frac{3t_g}{T^3 - \delta^3}\boldsymbol{m}_0, \qquad 0 \leq t \leq t_f \tag{26}$$

Therefore, the magnitude of $\boldsymbol{a}_{MO}$ is always finite and has its maximum at $t = 0$, and this magnitude decreases linearly after that until $t = t_f$. Note that $\boldsymbol{a}_{MO}$ is zero for $t > t_f$ and the magnitude of $\boldsymbol{a}_{MO}$ has a discontinuity at $t = t_f$ (which is the burnout time). Also the maximum magnitude of $\boldsymbol{a}_{MO}$, which occurs at $t = 0$, is proportional to the magnitude of the zero-effort miss vector at this time.

Moreover, explicit computation shows that the relative position and velocity at $t = t_f$ are given by

$$\boldsymbol{r}(t_f) = \delta\boldsymbol{p}_2 \tag{27}$$

$$\boldsymbol{v}(t_f) = -\boldsymbol{p}_2 \tag{28}$$

Therefore, the relative velocity is colinear with the relative position at $t = t_f$ [i.e., $\boldsymbol{r}(t_f) \times \boldsymbol{v}(t_f) = \boldsymbol{0}$], and these two vectors will remain colinear thereafter since the relative velocity is constant after t_f. In other words, this guidance law drives the LOS rate exactly to zero at $t = t_f$ (although the commanded acceleration is nonzero at this time), and this LOS rate remains zero thereafter and a perfect intercept at $t = T$ will occur since

$$\boldsymbol{r}(T) = \boldsymbol{r}(t_f) + \delta\boldsymbol{v}(t_f) = \delta\boldsymbol{p}_2 - \delta\boldsymbol{p}_2 = \boldsymbol{0} \tag{29}$$

where we have used the relations given in Eqs. (27) and (28).

It will be illuminating to compare the properties of this guidance law with those of the PN guidance law. Remember that in PN with a navigation constant of N, the acceleration command $\boldsymbol{a}_{\text{PN}}$ is computed through the following guidance law:

$$\boldsymbol{a}_{\text{PN}} = \frac{N}{t_g^2}[\boldsymbol{r}(t) + t_g\boldsymbol{v}(t)] \tag{30}$$

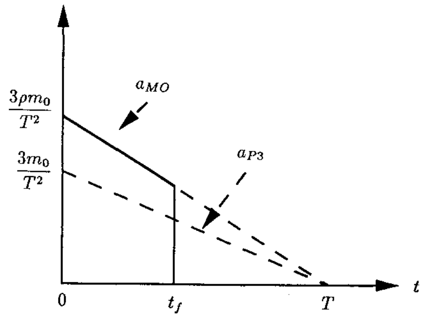

Fig. 1 Comparison of magnitude of acceleration command of optimal guidance law and PN for $N = 3$.

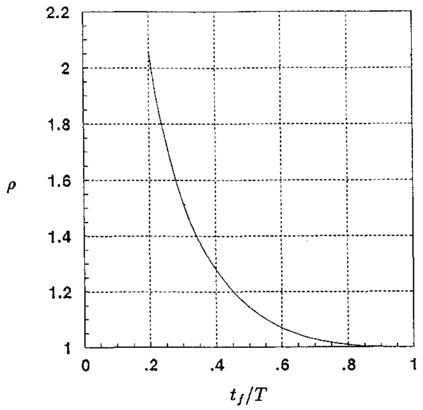

Fig. 2 Variation of ratio ρ as function of t_f/T.

Substituting a_{PN} for a_M in Eq. (3) and solving the resulting differential equation, it is simple to show that the acceleration command in this case has the form

$$a_{\text{PN}}(t) = -\ddot{r}(t) = \frac{N t_g^{N-2}}{T^N} m_0, \qquad t < T \tag{31}$$

where we have assumed momentarily that it is possible to execute acceleration maneuvers until intercept time T. In Fig. 1 we have compared the magnitude of the acceleration command of the optimal guidance law with that of PN with $N = 3$. By comparing Eqs. (26) and (31) for $N = 3$, it is clear that during the period $0 \leq t \leq t_f$ the magnitude of the acceleration command of the optimal guidance law is proportional to magnitude of the acceleration command of the PN with a navigation constant $N = 3$. In other words we have $a_{MO}(t) = \rho a_{P3}(t)$ with

$$\rho = \frac{1}{1 - (1 - t_f/T)^3} \tag{32}$$

Note that ρ depends only on the ratio t_f/T and this dependence is shown in Fig. 2.

Now let us assume that we use the PN guidance law until burnout, and after that we set the acceleration command to zero. In other words we set a_{PN} given in Eq. (31) equal to zero for $t > t_f$. It can be shown that the final miss vector $r(T)$ in this case will have the form

$$r(T) = \left(1 - \frac{t_f}{T}\right)^N m_0, \qquad 0 \leq t_f \leq T \tag{33}$$

where as before m_0 is the zero-effort miss vector at time zero. Hence, the miss distance is proportional to the magnitude of the zero-effort miss vector and the proportionality constant is a function of the ratio of t_f to T. Moreover, by increasing the navigation constant N, the magnitude of the final miss vector will be reduced; however, there is always a nonzero miss for finite N and $t_f < T$. Of course, increasing the navigation constant in PN increases the acceleration requirements of the missile, which is undesirable.

Now let us determine the navigation constant N of PN in such a way that the maximum value of the commanded acceleration in PN is the same as the maximum acceleration commanded by the optimal guidance law. This is accomplished by simply equating Eqs. (26) and (31) at $t = 0$. After performing the required substitutions we have

$$N = \frac{3T^3}{T^3 - \delta^3} \tag{34}$$

Substituting this value of N in Eq. (33), we can compute the miss distance caused by that PN guidance law that commands the same maximum acceleration as the proposed optimal guidance law:

$$r(T) = m_0 \left(\frac{\delta}{T}\right)^{3T^3/(T^3 - \delta^3)} \tag{35}$$

Here m_0 and $r(T)$ denote the magnitudes of the vectors m_0 and $r(T)$, respectively. (In general, we denote the magnitude of an arbitrary vector p by p.)

Motivated by the similarities between the proposed guidance law and PN for the case $N = 3$, it may be conjectured whether it is possible to generalize the optimal guidance law given in Eq. (20) to a more general form that guarantees a perfect intercept although not necessarily minimizing any particularly useful objective function. As a matter of fact it can be shown that for the guidance law

$$a_{MN}(t) = \frac{N t_g^{N-2}}{t_g^N - \delta^N} [r(t) + t_g v(t)] \tag{36}$$

we have

$$r(t_f) = \frac{N T^{N-1}\delta - N\delta^N}{(N-1)(T^N - \delta^N)} m_0 - \delta v_0 \tag{37}$$

$$v(t_f) = \frac{N\delta^{N-1} - N T^{N-1}}{(N-1)(T^N - \delta^N)} m_0 + v_0 \tag{38}$$

Hence the relative position and velocity are colinear at $t = t_f$ and $r(t_f) + \delta v(t_f) = 0$ so an intercept at $t = T$ will occur.

Note that the guidance law given in Eq. (20) for the case $\gamma \to \infty$ can be obtained by setting $N = 3$ in Eq. (36) ($a_{MO} = a_{M3}$). Also the PN guidance law given in Eq. (30) is obtained from Eq. (36) by setting $\delta = 0$.

Substituting Eq. (36) in Eq. (3) and solving the resulting differential equation, it can be shown that the acceleration $a_{MN}(t)$ will have the form

$$a_{MN}(t) = \frac{N t_g^{N-2}}{T^N - \delta^N} m_0, \qquad 0 \leq t \leq t_f \tag{39}$$

with $a_{MN}(t)$ assumed zero for $t > t_f$. Referring to this relation, if $N > 2$, then the magnitude of the commanded acceleration is monotonically decreasing during the period $0 \leq t \leq t_f$. The astonishing fact is that the guidance law given in Eq. (36) achieves a perfect intercept for any given initial conditions r_0 and v_0 and for all nonzero values of N if $\delta > 0$. However, the maximum acceleration command is quite large if $N < 2$ and δ is small compared to T.

But the guidance law in Eq. (36) will perform poorly if there is a target maneuver. Of course this is expected since there is a period of time that the missile cannot maneuver, and if the missile does not have any knowledge of the target acceleration, then it is impossible to guarantee the target intercept. However, in the presence of a constant target thrust maneuver a_T, it can be shown that the following guidance law will achieve an intercept:

$$a_{MN} = \frac{N t_g^{N-2}}{t_g^N - \delta^N} \left[r(t) + t_g v(t) + \tfrac{1}{2} t_g^2 a_T\right] \tag{40}$$

Here, again, the term in the bracket is the zero-effort miss vector.

Moreover, it can be shown that the guidance law in Eq. (40) with $N = 3$ minimizes the objective function given in Eq. (5) assuming the constant target thrust maneuver a_T is known and $1/\gamma = 0$. For the case of finite γ, simply add the constant term $3/\gamma$ to the denominator of the coefficient in Eq. (40) while setting $N = 3$.

IV. Actual Implementation

Actual implementation of the guidance law given in Eq. (20) for the three-dimensional case requires a few minor modifications, as we shall indicate shortly. First, note that for computing $\boldsymbol{a}_{MO}$ we need to measure (or estimate) the relative position and velocity. We assume this information is provided by a combination of external target tracking, missile self-navigation (or tracking), and state estimation.

Moreover, we require estimates for time to go until intercept t_g and time to missile burnout t_b. If the burning period of the rocket motor t_f is known accurately and the maneuvering starts right after the motor ignition, then t_b can be computed using its definition $t_b = t_f - t$ where t denotes the present time.

Another approach to estimating t_b is based on using the known total impulse ΔV of the missile. For this we integrate the actual thrust acceleration of the missile and use it in the following relation:

$$t_b = \frac{\Delta V - \int_0^t a_M(\tau)\,\mathrm{d}\tau}{a_T(t)} \tag{41}$$

Here, a_M denotes the magnitude of the thrust acceleration vector of the missile $\boldsymbol{a}_M$ (which we assume is measured). Note that Eq. (41) is an exact estimate if a_M is constant from current time to burnout. Otherwise this is an approximation whose accuracy improves as we get closer and closer to the burnout. Simulation results indicate that this approximation is quite adequate for our purpose. If necessary, we can even use a weighted average of $t_f - t$ (assuming t_f is known with reasonable accuracy) and t_b given in Eq. (41). Moreover, special care should be taken to make sure that the computed t_b is always positive and within reasonable bounds.

Also note that we have used $\boldsymbol{a}_M$ to denote the missile thrust acceleration vector, which is different from the optimal acceleration command a_{MO} computed using Eq. (20). The difference is because we may be forced to command a nonzero acceleration along the LOS, since we may not have any control on the magnitude of this missile acceleration. We will clarify the point shortly.

For computing the time to go t_g, let us denote the length of the vector $\boldsymbol{r}$ by r (which is the same as the missile-target distance). Then, an estimate for t_g is obtained using

$$t_g = -\frac{r}{\dot{r}} \tag{42}$$

Note that the term $\dot{r}$ in Eq. (42) is computed using

$$\dot{r} = \frac{1}{r}\boldsymbol{v}^T\boldsymbol{r} \tag{43}$$

Also there is an important property that deserves some attention. If we compute t_g using relation (42), then the zero-effort miss vector appearing inside the bracket in Eq. (20) is perpendicular to the LOS. Therefore, in this case the computed acceleration command $\boldsymbol{a}_{MO}(t)$ is perpendicular to the LOS.

The estimate given in Eq. (42) is exact if $\dot{r}$ is constant, but this is not the case in our problem. However, the accuracy improves as we get closer and closer to the time of missile burnout. A better estimate for t_g can be obtained by using the value of $\ddot{r}$. It is simple to show that if $\ddot{r}$ is constant during the burn period, an exact relation for t_g is as follows:

$$t_g = \frac{t_b^2\ddot{r}(t)/2 - r(t)}{t_b\ddot{r}(t) + \dot{r}(t)} \tag{44}$$

Even if $\ddot{r}$ is not constant, this relation leads to a good approximation of t_g. For small values of t_b or $\ddot{r}$, Eqs. (42) and (44) are almost identical. To use Eq. (44), we need to compute $\ddot{r}$. For this we use the identity

$$\ddot{r} = \frac{1}{r}\dot{\boldsymbol{v}}^T\boldsymbol{r} + \frac{1}{r}(v^2 - \dot{r}^2) \tag{45}$$

Noting $\dot{\boldsymbol{v}} = -\boldsymbol{a}_M$, the first term on the right-hand side of Eq. (45) is the negative of the component of the thrust acceleration vector along the LOS.

Extensive simulations indicate that, for best performance, t_g appearing inside the bracket in Eq. (20) should be computed using Eq. (42) and t_g appearing in the coefficient outside the bracket should be computed using Eq. (44).

In computing t_g and t_b, a check is made to see whether the computed t_g is larger than the computed t_b (i.e., the intercept actually occurs after burnout). If this is not the case, we assign the value of t_g to t_b.

Another important practical constraint is that usually there is no control on the magnitude of the thrust vector, and the only means of control is the direction of this vector. The function of the guidance law is to generate the desired direction of the thrust vector, and the autopilot rotates the missile so its thrust vector (approximately the missile axial direction) is aligned with this commanded direction.

We will compute the direction of the desired thrust vector in such a way that, if the missile thrust is aligned with this vector, the component of missile acceleration perpendicular to the LOS is equal to $\boldsymbol{a}_{MO}$ computed from Eq. (20). For this, we first check to make sure that the magnitude of acceleration command computed using Eq. (20) is within the maneuvering capability of the missile. This is done by checking the validity of the inequality

$$a_{MO} \leq \alpha a_M \tag{46}$$

where α is a design factor close to 1 (usually 0.9) and a_M is the magnitude of the thrust acceleration of the missile. If Eq. (46) does not hold, we scale $\boldsymbol{a}_{MO}$ as

$$\boldsymbol{a}_{MO}^{\text{new}} = \frac{\alpha a_M}{a_{MO}}\boldsymbol{a}_{MO} \tag{47}$$

Note that the value of α can be used as a design parameter to limit the attitude maneuver of the missile whenever necessary. From now on we assume that the acceleration command vector is scaled properly so its magnitude is within the achievable bounds and the same symbol $\boldsymbol{a}_{MO}$ is used to denote $\boldsymbol{a}_{MO}^{\text{new}}$.

Now let the component of the thrust acceleration vector $\boldsymbol{a}_T$ along the LOS from the missile to the target be denoted by β. We want to compute β in such a way that the magnitude of $\boldsymbol{a}_M$ is equal to the instantaneous measured acceleration of the missile $\boldsymbol{a}_M$. For this write

$$\boldsymbol{a}_M = \boldsymbol{a}_{MO} + \frac{\beta}{r}\boldsymbol{r} \tag{48}$$

Now take the dot product of Eq. (48) by itself, and assuming $\boldsymbol{a}_{MO}$ is perpendicular to the LOS, we have

$$a_m^2 = a_{MO}^2 + \beta^2 \tag{49}$$

Solving Eq. (49) for β we have

$$\beta = \sqrt{a_M^2 - a_{MO}^2} \tag{50}$$

Note that the term under the square root is positive because of the constraint we imposed on the magnitude of a_{MO}. We also only use the positive solution of β in Eq. (49) because otherwise the component of the missile thrust acceleration along the LOS will decrease the closing velocity.

Finally, we substitute the computed β in Eq. (48) and command the direction of the resulting $\boldsymbol{a}_M$ to the attitude autopilot of the missile.

V. Simulation Results

We implemented this guidance law in the System Build[5] environment using a simple point-mass flat-Earth simulation to analyze the performance of the proposed guidance law.

As an example trajectory, illustrating the effect of the initial heading error, we assumed the missile is at the origin of the coordinate frame at the beginning of the engagement and the components of the initial missile velocity (in meters per second) are

$$\boldsymbol{v}_M(0) = \begin{pmatrix} 600 \\ 150 \\ 0 \end{pmatrix} \tag{51}$$

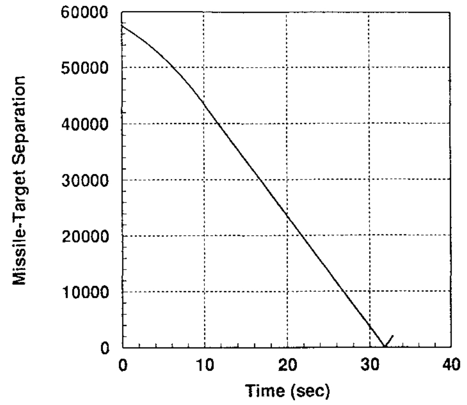

Fig. 3 Missile-target separation.

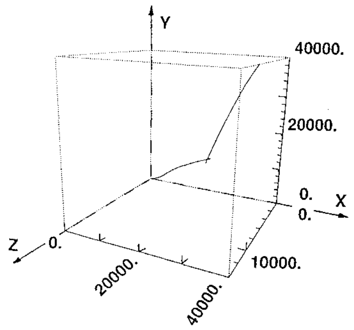

Fig. 4 Missile-target intercept geometry.

We have assumed that the y axis is in the vertical direction and the other two axes x and z are in the horizontal plane. The initial position (in meters) and velocity (in meters per second) of the target are

$$r_T(0) = \begin{pmatrix} 40{,}000 \\ 40{,}000 \\ 10{,}000 \end{pmatrix}, \qquad v_T(0) = \begin{pmatrix} -200 \\ -400 \\ 100 \end{pmatrix} \tag{52}$$

For purposes of illustration, the initial total mass of the missile is assumed 1000 kg and the constant propellant mass burning rate is assumed to be 35 kg/s. The total propellant mass is 350 kg. The specific impulse of the missile is assumed to be 270 s. This results in a total ΔV of 1140.55 m/s. However, we used a value of 1152 m/s in Eq. (41), which is 1% higher than the true value, to illustrate robustness to parameter uncertainty.

Moreover, we set the value of α in Eq. (47) equal to 0.7 to make sure that the angle between the desired direction of the thrust and the LOS is never larger than $\sin^{-1}$ (0.7) rad. We also set $1/\gamma = 1$ and added random noises with standard deviations of 20 and 0.5 m/s, respectively, to the position and velocity of the target and the missile.

Figure 3 shows the missile-target separation as a function of time. The time of closest passage occurs at 31.8 s, and the missile-target separation at this time is 16 m. Note that the actual burnout occurs at $t = 10$ s, and for a period of approximately 21.8 s before the intercept the missile is not maneuvering. In this simulation, the missile acceleration command is computed once every 0.5 s, and also a perfect autopilot is assumed. The three-dimensional missile-target engagement is shown in Fig. 4.

Moreover, we have used Eq.(41) for estimating t_b. For computing t_g appearing inside the bracket in Eq. (20), we used Eq. (42). But the t_g appearing in the fraction outside of the bracket was computed using Eq. (44). The behavior of the estimated t_b and t_g in this simulation is shown in Fig. 5. Note that after burnout any call to the guidance block is disabled and the value of t_g is not updated anymore. It is clear that after the initial transient the estimate given by Eq. (44) is very close to the exact value.

The angle between the direction of the missile thrust vector (missile longitudinal axis) and the missile velocity vector is shown in Fig. 6. Note that after missile burnout we have set this angle to zero. In this particular simulation, an acceleration saturation occurred only during the first guidance interval (first 0.5 s), and the scaling given in Eq. (47) was enforced automatically. As a matter of fact, if we had set $\alpha = 0.9$ (as we have proposed), the scaling (47) would not have been required for this particular initial condition.

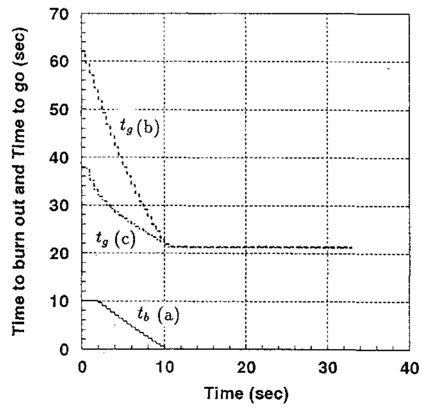

Fig. 5 a) Estimate of time to burnout t_b, b) estimate of t_g using Eq. (42), and c) estimate of t_g using Eq. (44).

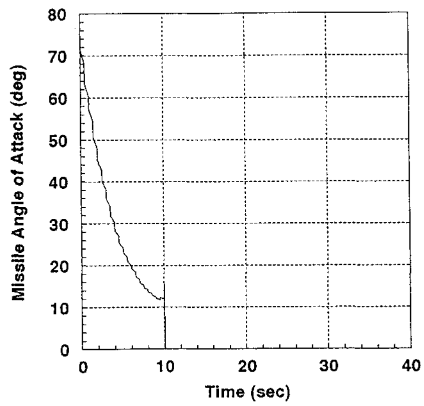

Fig. 6 Angle between missile thrust and its velocity vector.

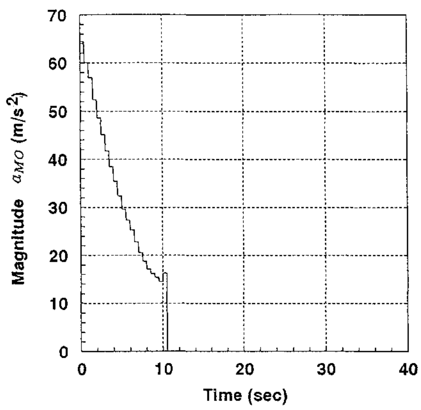

Fig. 7 Magnitude of computed a_{MO}.

The magnitude of the computed a_{MO} at each guidance interval is shown in Fig. 7. Note that because of the perturbation error on the total impulse, the guidance law issues a command after missile burnout that is just an artifact of the simulation and has no effect on the missile trajectory since there is no maneuvering capability available after $t = 10$.

Also, to make sure that at burnout the missile has actually been placed on a collision course with the target, we computed the instantaneous magnitude of the LOS rate. This was computed by dividing the magnitude of the component of the relative velocity perpendicular to the LOS by the missile-target distance and is shown in Fig. 8.

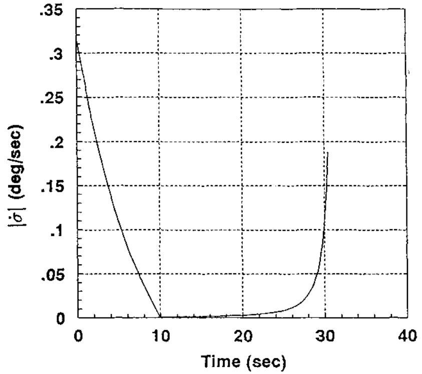

Fig. 8 Magnitude of LOS rate.

As we expected, at burnout ($t = 10$), the LOS rate is driven to very small values. The residual LOS rate that remains is of course due to different error sources. Also note that we have shown the LOS rate only during the first 31 s of the trajectory, because for small missile-target distances, small miss distances generate large LOS rates that completely overshadow the behavior of this quantity during the rest of the trajectory.

To further analyze the performance of the guidance law, we repeated the simulation for several different (reasonable) initial conditions. In all the simulations an error of about 1% on the knowledge of ΔV was assumed and the standard deviation of the error in the knowledge of the missile and target position and velocity were respectively taken to be 20 and 0.5 m/s. In all the simulations, the miss distance was less than 90 m, with majority of them being less than 50 m. This clearly illustrates the performance of this guidance law.

VI. Conclusion

In this paper, we formulated the problem of guiding a missile with time-limited maneuvering capability to intercept a nonmaneuvering target. We first computed the optimal guidance law for this class of problems and later analyzed the properties of the optimal solution in detail. We also generalized this guidance law to a family of guidance laws that guarantee a perfect intercept in the case of finite-time maneuvers. Moreover, we discussed several practical constraints that are involved in implementing such a guidance law and illustrated its performance using a simple example.

There are other details that we have not discussed here. For example, the effect of autopilot dynamics and sensor noise on the performance of this guidance law should be studied. However, the similarities between the guidance law proposed here and the PN guidance law should help us in this regard.

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